Eur. Phys. J. Plus (2024) 139:974
https://doi.org/10.1140/epjp/s13360-024-05765-0

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

New diverse types of the soliton arising from the integrable Kuralay equations against its numerical solutions

Emad H. M. Zahran[1,a], **Zhanar Umurzakhova**[2], **Ahmet Bekir**[3,b], **Ratbay Myrzakulov**[2,c], **Reda A. Ibrahim**[1,d]

[1] Department of Basic Science, Faculty of Engineering, Benha University, Shubra, Egypt
[2] Eurasian International Centre for Theoretical Physics, Astana, Kazakhstan
[3] Neighbourhood of Akcaglan, Imarli Street, Number: 28/4, 26030 Eskisehir, Turkey

Received: 21 March 2024 / Accepted: 21 October 2024

Abstract The paper aims to establish diverse types of the soliton solutions for the integrable Kuralay equations to discuss the integrable motion of the induced space curves by these equations. The solitons arising from the integrable Kuralay equations are considered by tall superiority and qualitative studies for many effective phenomena in various fields such as ferromagnetic materials, nonlinear optics and optical fibers. There are two various schemes are suggested to establish these diverse types of solitons that arise from this model, namely the extended simple equation method and the Paul-Painleve approach method. New diverse types of the soliton solutions that appear in forms of periodic trigonometric function soliton solutions, parabolic function soliton solutions, singular soliton solutions, W-like soliton solutions and M-like soliton solutions have been documented. The suggested techniques are used for the first time for this target. The achieved soliton solutions will offer a rich podium to study the nonlinear spin dynamics in magnetic materials. Moreover, we will construct the numerical solutions for all achieved soliton solutions by using the differential transform methods. The comparison between the new achieved soliton solutions with its consistent numerical solutions has been documented.

Abbreviations

IKE	Integrable Kuralay Equation
ESEM	Extended Simple Equation Method
PPAM	Paul-Painleve Approach Method
DTM	Differential Transform Methods

1 Introduction

We will focus on the integrable soliton equations that are the most significant class of NLPDE have remarkable applications to many applied mathematics and physical systems, such as fluid mechanics, nonlinear optics, plasma physics, hydrodynamics, field theory, etc. The nonlinear integrable system that has an extremely vital mission in contemporary mathematical physics applications is characterized by numerous types: solitons, Lax pairs, Painleve tests, etc. Recently, numerous influential mathematical techniques such as the inverse scattering transform, the Hirota bilinear, the Bell polynomial approximate, the Darboux transformation, Painleve analysis and so on are proposed to extract the exact traveling wave solutions of the integrable nonlinear equations. In the other hand, there are other mathematical methods that are recently discovered to extract the exact traveling wave solutions for NLPDE, some of them surrender to the balance rule as the (G'/G)-expansion method [1–4], the extended simple equation method (ESEM) [5–7], the extended direct algebraic method (EDAM) [8–10], the modified extended mapping method (MEMM) [11], the Paul-Painleve approach method [12–15], the modified extended tanh function method [16, 17], the modified simple equation method [18, 19], while there exist only two methods that don't surrender to the balance rule, namely the solitary wave ansatz method [20, 21] and the Riccati-Bernoulli-Sub OD method [22–24]. There are many trials to study the soliton solutions for various nonlinear problems arising in different branches of science have been demonstrated through [25–40]. Our study specially will focus on the (1 + 1) and (2

[a] e-mail: e_h_zahran@hotmail.com
[b] e-mail: bekirahmet@gmail.com (corresponding author)
[c] e-mail: rmyrzakulov@gmail.com
[d] e-mail: reda.mohamed@feng.bu.edu.eg

+ 1) Heisenberg ferromagnetic-type dimensions' equations that are representing the integrable spin systems. The (2 + 1) Heisenberg ferromagnetic-type has been studied before [41–43]. Moreover, some important studies for the soliton in this field have been studied, see for example Seadawy, et al. [44] who applied the extended direct algebraic method to the Zakharov–Kuznetsov equation to find the electric field potential, electric field and magnetic field in the form of traveling wave solutions, Seadawy, et al. [45] who introduced, analyzed recently the new three-dimensional modified Benjamin-Bona-Mahony equations with the introduction of the spatial and temporal fractional order derivatives using conformable fractional derivative and constructed various solutions including hyperbolic and periodic function solutions by using simple ansatzs with the aid of the Wolfram Mathematica software, Seadawy, et al. [46] who computed the conservation laws of Chen–Lee–Liu equation with the help of scaling invariance technique, used Euler and Homotopy operators for the evaluation of conserved densities and fluxes and obtained domain walls, solitary wave and elliptic wave solutions with the help of Unified method, Seadawy, et al. [47] who studied weakly nonlinear wave propagation theory in the occurrence of magnetic fields in fluids, constructed the soliton and other kinds solutions of (2 + 1)-dimensional elliptic nonlinear Schrödinger equation by using the modified and extended rational expansion method and obtained different kinds of wave solutions such as bright-dark solitons, solitons of Kink and anti-Kink, solitary waves, periodic solutions and elliptic function solutions of this equation, Jhangeer, et al. [48] who obtained diverse range of traveling wave structures of perturbed Fokas–Lenells model by using the extended (G'/G^2)-expansion technique and used the Runge–Kutta fourth-order technique to extract the nonlinear periodic solutions of the considered problem, Iqbal, et al. [49] who used the auxiliary equation mapping method and direct algebraic method to drive new exact and solitary wave solutions in the form of trigonometric function, periodic solitary wave, rational function and elliptic function, hyperbolic function, bright and dark solitons solutions for the nonlinear longitudinal wave equation which involves mathematical physics with dispersal produced by the phenomena of transverse Poisson's effect in a magneto-electro-elastic circular rod and Wang, et al. [50] who used the two-variable (G′/G,1/G)-expansion approach to a new coupled KdV and Z-K system, which has various significant applications in different fields of applied sciences and constructed different forms of analytical solutions of the new coupled KdV system and the new coupled system, such as solitons, multi-peak solitons, solitary waves, trigonometric, hyperbolic and rational functions and other wave solutions. In the same connection, there are recent works for studying various fractional nonlinear problems arising in various branches of science, see for example, Nasreen, et al. [51] who investigated the dynamics of optical waves to the generalized coupled nonlinear fractional Helmholtz equation with quantic and cubic nonlinear effects and extracted different types of the solutions like bright, dark, singular and mixed type solitons by applying advanced integration methods, namely modified Sardar sub-equation method and new Kudryashov method, Younas, et al. [52] who used the modified generalized Riccati equation mapping method and the generalized exponential rational function method approach to study the comparative exact solutions of the Kairat-II equation that describing the physical behaviors of nonlinear systems and has numerous applications in the fields of plasma physics, optical communications, differential geometry engineering, oceanography and physics, Muhammad, et al. [53] who used the generalized Arnous method and multivariate generalized exponential rational integral function approach to explore the soliton wave dynamics of the fractional three-component coupled nonlinear Schrödinger equation that describing the behavior of optical pulses in optical fibers, Nasreen, et al. [54] who used the new extended direct algebraic method to obtain the bright, dark, combo and singular soliton solutions to the higher order nonlinear Schrödinger equation with the effect of group velocity dispersion, second-order spatiotemporal dispersion and cubic nonlinearity that describing the optical pulses propagation in different forms in the nonlinear optical fiber and Younas, et al. [55] who utilized the new extended direct algebraic approach to derive various forms including bright, dark and combo solitons to the third-order nonlinear fractional Westervelt model that is advantageous in the examination of sound wave propagation and high amplitude phenomena in the fields of medical imaging and therapy.

The integral equations are important in many applications. Problems in which integral equations are encountered include radiate transfer and the oscillation of a string, membrane or axle. The most obvious advantage of methods based on integral equations arises for problems involving constant coefficient differential operators (i.e., physical properties such as conductivity or wave speed remain constant throughout the domains) without nobody loads. The integrable Kuralay equation is one of the famous integral equations in physics that related with many integrable spin systems. The integrable Kuralay equation that related with many integrable spin systems can be proposed according to [56–61] as

$$\begin{aligned} &iH_t - H_{xt} - vH = 0,\\ &iR_t + R_{xt} + vR = 0,\\ &v_x + 2d^2(RH)_t = 0. \end{aligned} \tag{1}$$

where $H(x, t)$ is complex function whose complex conjugate is $\overline{H}(x, t)$, , while $v(x, t)$ is a potential real function, x, t are the independent spatial and temporal variables.

In soliton theory, the gauge and geometrical equivalence play a fundamental role. There are two versions of the integrable Kuralay equation (K-IIE) which are (K-IIAE) AND (K-IIBE).

Let $d = 1$, $R = \varepsilon\overline{H}$ where $\varepsilon = \pm 1$ then the above system will be converted to

$$\begin{aligned} &iH_t - H_{xt} - vH = 0,\\ &v_x - 2\varepsilon(|H|^2)_t = 0 \end{aligned} \tag{2}$$

Now, let us introduce the transformation

$$H(x,t) = U(\zeta)e^{i\,(kx+wt+\eta)}, \zeta = mx + ct \tag{3}$$

Hence,

$$H_t = (cU' + iwU)e^{i(kx+wt+\eta)} \tag{4}$$

$$H_x = (mU' + ikU)e^{i(kx+wt+\eta)} \tag{5}$$

$$H_{xt} = (cmU'' + iwmU' + ickU' - kwU)e^{i(kx+wt+\eta)} \tag{6}$$

According to the above relations (3–6), the system Eq. (2) becomes

$$\begin{aligned} &i\big(cU' + iwU\big) - \big(cmU'' + iwmU' + ickU' - kwU\big) - vU = 0, \\ &mv' - 4\varepsilon cU' = 0 \end{aligned} \tag{7}$$

By integrating the second part of Eq. (7), we get

$$v = \frac{2\varepsilon cU^2}{m} - \frac{c_1}{m} \tag{8}$$

By inserting Eq. (8) in the first part of Eq. (7), we obtain

$$i\big(cU' + iwU\big) - \big(cmU'' + iwmU' + ickU' - kwU\big) - \left(\frac{2\varepsilon cU^2}{m} - n\right)U = 0. \tag{9}$$

Equation (9) implies the following real and imaginary parts

$$U'' + \frac{(w(1-k)-n)}{cm}U + \frac{2\varepsilon U^3}{m^2} = 0 \tag{10}$$

$$(c - wm - ck)U' = 0 \tag{11}$$

The imaginary part Eq. (11) implies that

$$m = \frac{c(k-1)}{w} \tag{12}$$

By inserting Eq. (12) into Eq. (10), we get

$$U'' + \frac{w(w(1-k)-n)}{c^2(k-1)}U + \frac{2w^2\varepsilon U^3}{c^2(k-1)^2} = 0 \tag{13}$$

Now our main aim is extracting new forms of the traveling wave solutions of Eq. (13) in the framework of two distinguished and effective of the ansatz techniques which are the extended simple equation method, the Paul-Painleve Approach Method. Moreover, to ensure the validity of the obtained soliton solutions and prove the consistency, we apply the efficient DTM-numerical method to derive the identical numerical solutions for all achieved soliton solutions. To implement the suggested techniques, we apply the homogeneous balance theory between U'', U^3 to get $N + 2 = 3N \Rightarrow N = 1$.

This research is prepared as follow; in first section, the introduction is considered, in the second section, the extended simple equation method (ESEM) and its applications to extract other perceptions to the traveling wave solutions of this model, in the third section, the Paul-Painleve approach method (PPAM) and its applications to extract other types of the traveling wave solutions to this model, in the fourth section, the differential transform method and its applications to derive the numerical solutions for the achieved soliton solutions and in the 6th section, the conclusion is presented.

2 The ESEM

To discuss this technique, let us overview the nonlinear partial differential equation form that can be introduced as

$$Q(U, U_x, U_t, U_{xx}, U_{tt}, \ldots) = 0 \tag{14}$$

where Q is a function of φ, its highest order partial derivatives and the nonlinear terms. When Eq. (14) surrenders to the transformation $U(x, t) = U(\zeta)$, $\zeta = mx + ct$ it will be converted to the following ODE:

$$\Upsilon\big(U, U', U'', \ldots\big) = 0. \tag{15}$$

where Υ in terms of $U(\zeta)$ and total derivatives with respect to ζ, the suggested solution of Eq. (13) according to the ESEM is:

$$U(\zeta)=\sum_{i=-N}^{N}A_i h^i(\zeta) \tag{16}$$

where $h(\zeta)$ appearing in Eq. (16) can be determined from the relation

$$h'(\zeta)=a_0+a_1h+a_2h^2+a_3h^3 \tag{17}$$

While the integer $N=1$ is previously determined.
That ESEM proposes these two families of solutions:

1. If $a_1=a_3=0$, then Eq. (17) represents the Riccati equation from which we get these solutions

$$h(\zeta)=\frac{\sqrt{a_0a_2}}{a_2}\tan(\sqrt{a_0a_2}(\zeta+\zeta_0),a_0a_2\succ 0 \tag{18}$$

$$h(\zeta)=\frac{\sqrt{-a_0a_2}}{a_2}\tanh(\sqrt{-a_0a_2}\zeta-\frac{\mu\ln\zeta_0}{2}),a_0a_2\prec 0,\zeta\succ 0,\mu=\pm 1 \tag{19}$$

2. If $a_0=a_3=0$, then Eq. (17) represents the Bernoulli equation from which we get these solutions

$$h(\zeta)=\frac{a_1Exp[a_1(\zeta+\zeta_0)]}{1-a_2Exp[a_1(\zeta+\zeta_0)]},a_1\succ 0 \tag{20}$$

$$h(\zeta)=\frac{-a_1Exp[a_1(\zeta+\zeta_0)]}{1+a_2Exp[a_1(\zeta+\zeta_0)]},a_1\prec 0 \tag{21}$$

and the general solutions to Eq. (17) are:

$$h(\zeta)=-\frac{1}{a_2}\left(a_1-\sqrt{4a_1a_2-a_1^2}\tan\left(\frac{\sqrt{4a_1a_2-a_1^2}}{2}(\zeta+\zeta_0)\right)\right),4a_1a_2\succ a_1^2,a_2\succ 0 \tag{22}$$

$$h(\zeta)=\frac{1}{a_2}\left(a_1+\sqrt{4a_1a_2-a_1^2}\tanh\left(\frac{\sqrt{4a_1a_2-a_1^2}}{2}(\zeta+\zeta_0)\right)\right),4a_1a_2\succ a_1^2,a_2\prec 0 \tag{23}$$

where ζ_0 is integration constancy.

I. *The solution according to the first family is*:

$$U(\zeta)=\frac{A_{-1}}{h}+A_0+A_1h \tag{24}$$

$$U'=-\frac{a_0A_{-1}}{h^2}+A_1a_0+A_1a_2h^2-a_2A_{-1} \tag{25}$$

$$U''=\frac{2a_0^2A_{-1}}{h^3}+\frac{2a_0a_2A_{-1}}{h}+2A_1a_0a_2h+2A_1a_2^2h^3 \tag{26}$$

Via inserting the relations (52–54) into the suggested model Eq. (13), equating the coefficients of various powers of h^i to zero, this implies a system of equations whose solution is:

$$A_0=A_1=0,A_{-1}=\frac{-ic(k-1)a_0}{w\sqrt{\varepsilon}},a_2=\frac{w(n+kw-w)}{2c^2(k-1)a_0} \tag{27}$$

$$A_0=A_1=0,A_{-1}=\frac{ic(k-1)a_0}{w\sqrt{\varepsilon}},a_2=\frac{w(n+kw-w)}{2c^2(k-1)a_0} \tag{28}$$

$$A_0=A_{-1}=0,A_1=\frac{-ic(k-1)a_2}{w\sqrt{\varepsilon}},a_0=\frac{w(n+kw-w)}{2c^2(k-1)a_2} \tag{29}$$

$$A_0=A_{-1}=0,A_1=\frac{ic(k-1)a_2}{w\sqrt{\varepsilon}},a_0=\frac{w(n+kw-w)}{2c^2(k-1)a_2} \tag{30}$$

$$A_0=0,A_{-1}=\frac{-ic(k-1)a_0}{w\sqrt{\varepsilon}},a_2=\frac{-w(n+kw-w)}{4c^2(k-1)a_0},$$
$$A_1=\frac{1}{ca_0\sqrt{\varepsilon}}\left\{\frac{3in^2-3ikn^2-6inw+12iknw-6ik^2nw+3iw^2-9ikw^2+9ik^2w^2-3ik^3w^2}{12n-12kn-8w+24wk-4w-12wk^2}\right\} \tag{31}$$

$$A_0 = 0, A_{-1} = \frac{ic(k-1)a_0}{w\sqrt{\varepsilon}}, a_2 = \frac{-w(n+kw-w)}{4c^2(k-1)a_0},$$
$$A_1 = \frac{-1}{ca_0\sqrt{\varepsilon}}\left\{\frac{\begin{array}{l}3in^2 - 3ikn^2 - 6inw + 12iknw - 6ik^2nw+\\ 3iw^2 - 9ikw^2 + 9ik^2w^2 - 3ik^3w^2\end{array}}{12n - 12kn - 8w + 24wk - 4w - 12wk^2}\right\} \tag{32}$$

$$A_0 = 0, A_{-1} = \frac{-ic(k-1)a_0}{w\sqrt{\varepsilon}}, a_2 = \frac{w(n+kw-w)}{4c^2(k-1)a_0},$$
$$A_1 = \frac{1}{ca_0\sqrt{\varepsilon}}\left\{\frac{\begin{array}{l}3in^2 - 3ikn^2 - 6inw + 12iknw - 6ik^2nw+\\ 3iw^2 - 9ikw^2 + 9ik^2w^2 - 3ik^3w^2\end{array}}{12n - 12kn - 8w + 24wk - 4w - 12wk^2}\right\} \tag{33}$$

$$A_0 = 0, A_{-1} = \frac{ic(k-1)a_0}{w\sqrt{\varepsilon}}, a_2 = \frac{w(n+kw-w)}{4c^2(k-1)a_0},$$
$$A_1 = \frac{-1}{ca_0\sqrt{\varepsilon}}\left\{\frac{\begin{array}{l}3in^2 - 3ikn^2 - 6inw + 12iknw - 6ik^2nw+\\ 3iw^2 - 9ikw^2 + 9ik^2w^2 - 3ik^3w^2\end{array}}{12n - 12kn - 8w + 24wk - 4w - 12wk^2}\right\} \tag{34}$$

These 8th–various results will generate 8th–various solutions, for similarity, we will construct the solutions of the first and fifth results only.

1. For the first result which is:

$$A_0 = A_1 = 0,\ A_{-1} = \frac{-ic(k-1)a_0}{w\sqrt{\varepsilon}},\ a_2 = \frac{w(n+kw-w)}{2c^2(k-1)a_0}$$

This result can be simplified to be

$$A_0 = A_1 = 0, A_{-1} = 1, a_0 = a_2 = -1 \tag{35}$$

Thus, the solution according to this result and the suggested method is (Figs. 1, 2, 3):

$$U(\zeta) = \frac{A_{-1}}{h} + A_0 + A_1 h \tag{36}$$

where $h(\zeta) = -\tan(-x - t - 1)$

$$U(\zeta) = -\cot(x + t + 1) \tag{37}$$

$$v(\zeta) = 2(\cot[x + t + 1])^2 - 1 \tag{38}$$

$A_0 = A_1 = 0,\ A_{-1} = 1,\ a_0 = a_2 = -1,\ k = 2,\ c = m = n = w = 1,\ \varepsilon = -1.$
$A_0 = A_1 = 0,\ A_{-1} = 1,\ a_0 = a_2 = -1,\ k = 2,\ c = m = n = w = 1,\ \varepsilon = -1.$

2. For the fifth result which is:

$$A_0 = 0,\ A_{-1} = \frac{-ic(k-1)a_0}{w\sqrt{\varepsilon}},\ a_2 = \frac{-w(n+kw-w)}{4c^2(k-1)a_0},$$
$$A_1 = \frac{1}{ca_0\sqrt{\varepsilon}}\left\{\frac{\begin{array}{l}3in^2 - 3ikn^2 - 6inw + 12iknw - 6ik^2nw+\\ 3iw^2 - 9ikw^2 + 9ik^2w^2 - 3ik^3w^2\end{array}}{12n - 12kn - 8w + 24wk - 4w - 12wk^2}\right\}$$

This result can be simplified to be

$$A_0 = 0, A_{-1} = 1, a_0 = -1, a_2 = 0.5, A_1 = -0.5, \varepsilon = -1, n = w = m = c = 1, k = 2 \tag{39}$$

Thus, the solution in the framework of these values is (Figs. 4, 5):

$$U(\zeta) = \frac{A_{-1}}{h} + A_0 + A_1 h$$

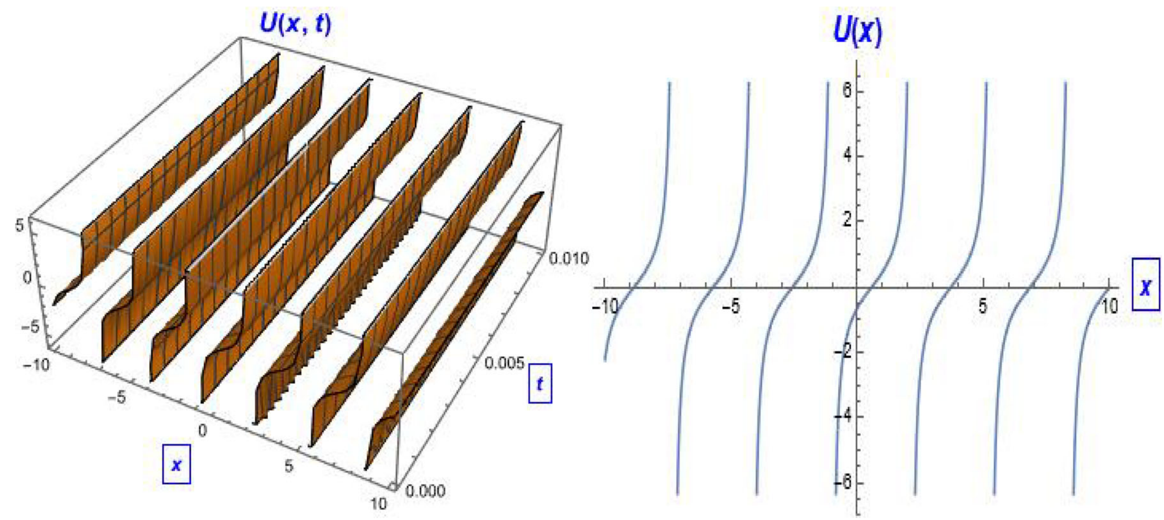

Fig. 1 The graph of $U(x, t)$ Eq. (37) in 2D and 3D with values

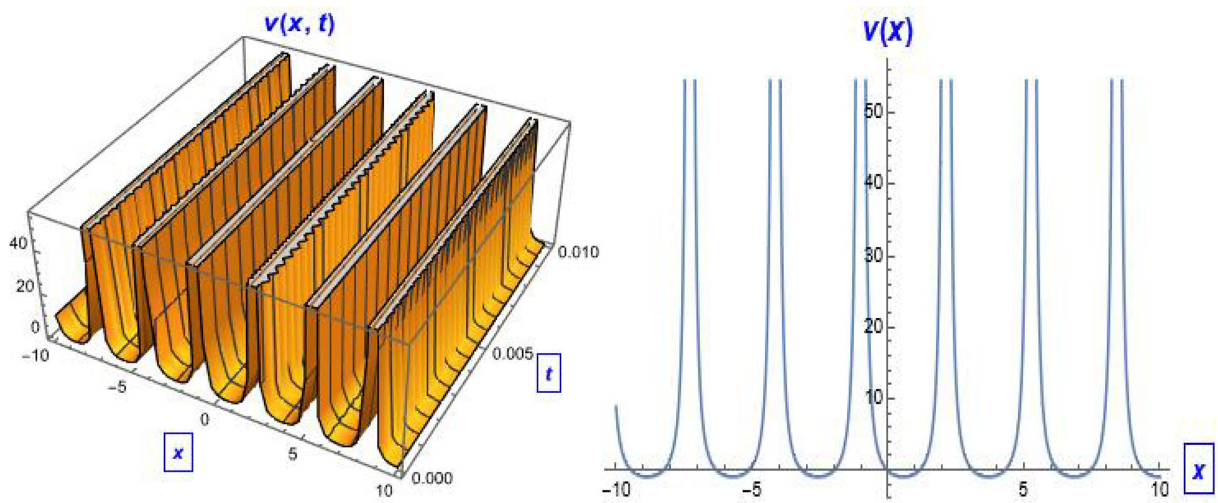

Fig. 2 The graph of $v(x, t)$ Eq. (38) in 2D and 3D with values

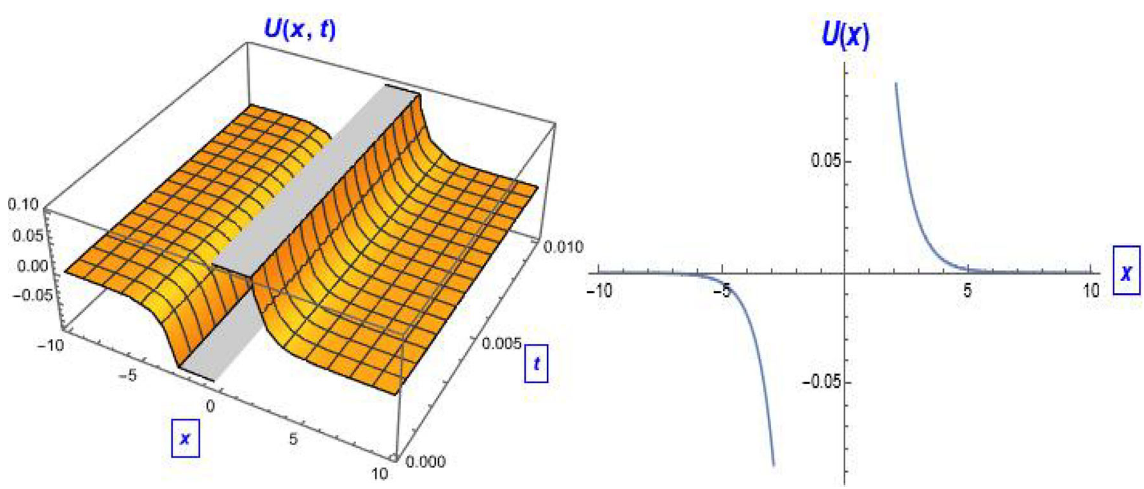

Fig. 3 The graph of $U(x, t)$ Eq. (40) in 2D and 3D with values

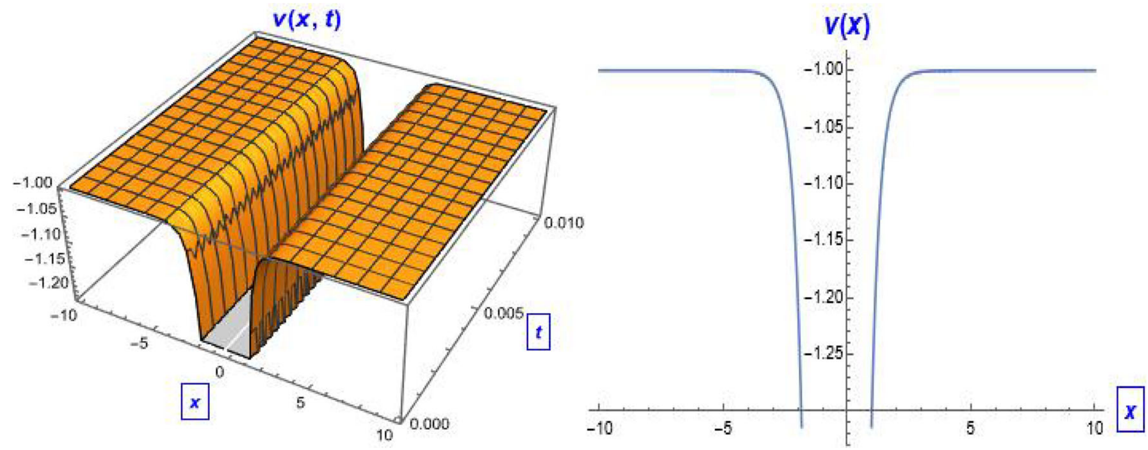

Fig. 4 The graph of $v(x, t)$ Eq. (41) in 2D and 3D with values

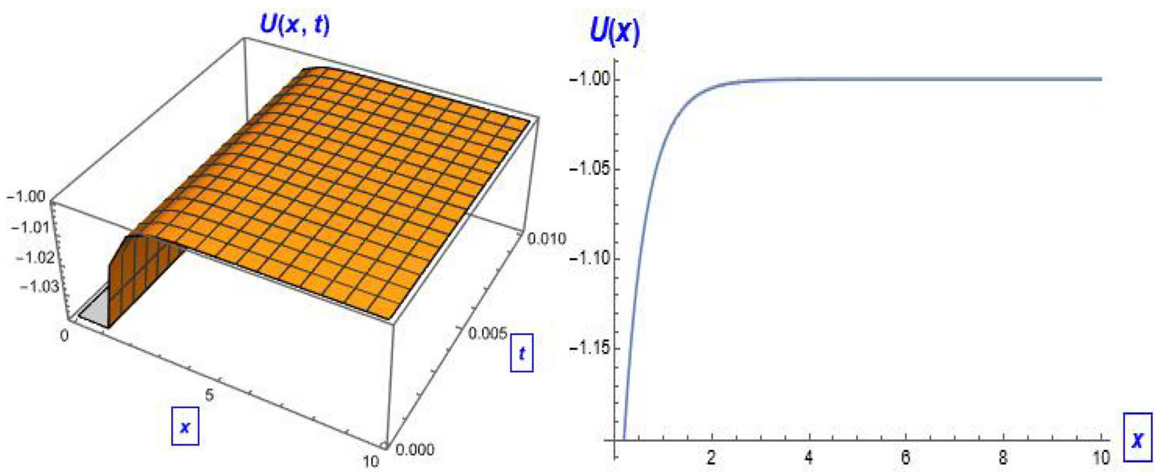

Fig. 5 The graph of $U(x, t)$ Eq. (48) in 2D and 3D with values

where $h(\zeta) = 1.4\tanh(0.7x + 0.7t + 0.3)$

$$U(\zeta) = 0.7\coth(0.7x + 0.7t + 0.3) - 0.7\tanh(0.7x + 0.7t + 0.3) \tag{40}$$

$$v(\zeta) = -2 * (0.7\coth[0.7x + 0.7t + 0.3] - 0.7\tanh[0.7x + 0.7t + 0.3])^2 - 1 \tag{41}$$

$A_0 = 0,\ A_{-1} = 1,\ a_0 = -1,\ a_2 = 0.5,\ A_1 = -0.5,\ \varepsilon = -1,\ n = w = m = c = 1,\ k = 2,\ \zeta_0 = 2,\ \mu = -1.$
$A_0 = 0,\ A_{-1} = 1,\ a_0 = -1,\ a_2 = 0.5,\ A_1 = -0.5,\ \varepsilon = -1,\ n = w = m = c = 1,\ k = 2,\ \zeta_0 = 2,\ \mu = -1.$
By the same manner, we can design the other solutions.

II *The solution according to the second family can be obtained as follow*

$$U' = A_1 a_2 h^2 + a_1 A_1 h - \frac{A_{-1} a_1}{h} - A_{-1} a_2 \tag{42}$$

$$U'' = 2A_1 a_2^2 h^3 + 3A_1 a_1 a_2 h^2 + A_1 a_1^2 h + A_{-1} a_1 a_2 + \frac{a_1^2 A_{-1}}{h} \tag{43}$$

Via inserting the relations (36) and (42–43) into the Eq. (13), equating the coefficients of various powers of h^i to zero, this implies a system of equations from which we get these results

$$A_{-1} = 0, A_0 = \frac{ic(k-1)a_1}{2w\sqrt{\varepsilon}}, A_1 = \frac{ic(k-1)a_2}{w\sqrt{\varepsilon}}, n = \frac{-(k-1)(2w^2 + c^2 a_1^2)}{2w} \tag{44}$$

$$A_{-1} = 0, A_0 = \frac{-ic(k-1)a_1}{2w\sqrt{\varepsilon}}, A_1 = \frac{-ic(k-1)a_2}{w\sqrt{\varepsilon}}, n = \frac{-(k-1)(2w^2 + c^2 a_1^2)}{2w} \tag{45}$$

These two various results will generate two various solutions from which we will construct the first one, which is

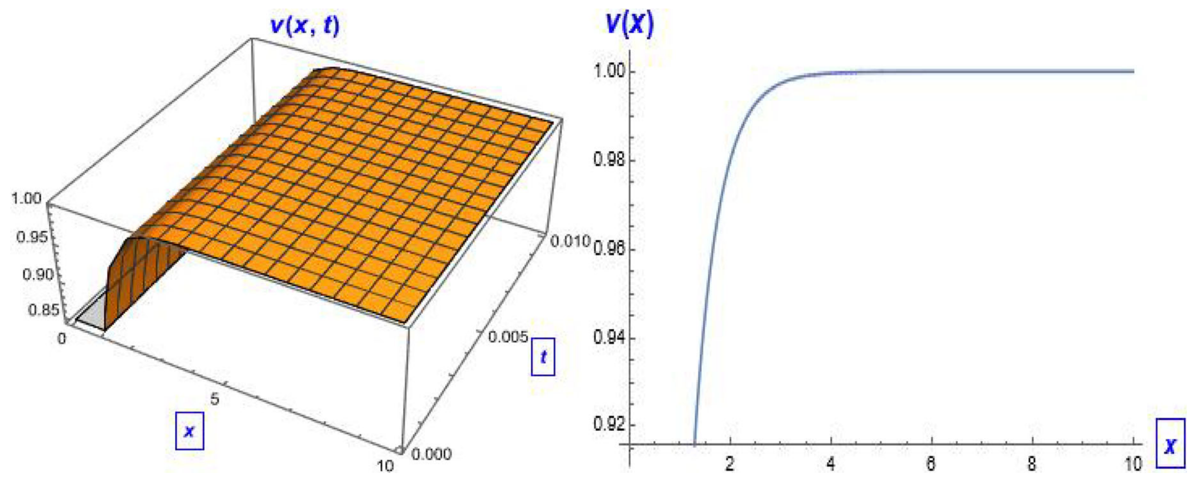

Fig. 6 The graph of $v(x,t)$ Eq. (49) in 2D and 3D with values

$$A_{-1}=0,\ A_0=\frac{ic(k-1)a_1}{2w\sqrt{\varepsilon}},\ A_1=\frac{ic(k-1)a_2}{w\sqrt{\varepsilon}},\ n=\frac{-(k-1)(2w^2+c^2a_1^2)}{2w}$$

This result can be simplified to be (Figs. 5, 6)

$$A_{-1}=0, A_0=A_1=1, k=2, w=c=1, \varepsilon=-1, a_1=2, a_2=1, n=-3 \tag{46}$$

Thus, the solution is $U(\zeta)=\frac{A_{-1}}{h}+A_0+A_1h$.
where $h(\zeta)=\frac{2Exp[2(\zeta+\zeta_0)]}{1-Exp[2(\zeta+\zeta_0)]}$

$$U(\zeta)=1+\frac{2Exp[2(\zeta+\zeta_0)]}{1-Exp[2(\zeta+\zeta_0)]} \tag{47}$$

$$U(\zeta)=\frac{1+Exp[2(\zeta+\zeta_0)]}{1-Exp[2(\zeta+\zeta_0)]} \tag{48}$$

$$v(\zeta)=-2\left(\frac{1+Exp[2(\zeta+\zeta_0)]}{1-Exp[2(\zeta+\zeta_0)]}\right)^2+3 \tag{49}$$

$A_{-1}=0,\ A_0=A_1=1,\ k=2,\ w=c=1,\ \varepsilon=-1,\ a_1=2,\ a_2=1,\ n=-3.$
$A_{-1}=0,\ A_0=A_1=1,\ k=2,\ w=c=1,\ \varepsilon=-1,\ a_1=2,\ a_2=1,\ n=-3.$
By the same manner, we can design the other solutions.

3 The PPAM

The PPAM introduces the solution for Eq. (13) whose balance is $N=1$ in the form

$$U(\zeta)=A_0+A_1W(\chi), \chi=R(\zeta)=D-\frac{e^{-N\zeta}}{N} \tag{50}$$

where $W(\chi)$ in Eq. (50) fulfills Riccati equation in the form $W_X-AW^2=0$ whose solution is:

$$W(\chi)=\frac{1}{A\chi+\chi_0}, \chi=R(\zeta)=D-\frac{e^{-N\zeta}}{N} \tag{51}$$

Via simple calculations for Eq. (50) we get

$$U'=-NA_1e^{-N\zeta}W-AA_1e^{-2N\zeta}W^2 \tag{52}$$

$$U''=N^2A_1e^{-N\zeta}W+3AA_1Ne^{-2N\zeta}W^2+2A^2A_1e^{-3N\zeta}W^3 \tag{53}$$

By introducing U'', U into Eq. (13) and equating the coefficients of various powers of $W^{iN}e^{-iN\zeta}$ to zero, this implies a system of equations from which the following results must be detected

$$A_0=\frac{-\sqrt{k-1}\sqrt{n+(k-1)w}}{\sqrt{2w\varepsilon}}, A=\frac{A_1\sqrt{w\varepsilon}\sqrt{-w(n+(k-1)w}}{c(k-1)\sqrt{n+(k-1)w}}, N=\frac{-\sqrt{2}\sqrt{-w(n+(k-1)w}}{c\sqrt{k-1}} \tag{54}$$

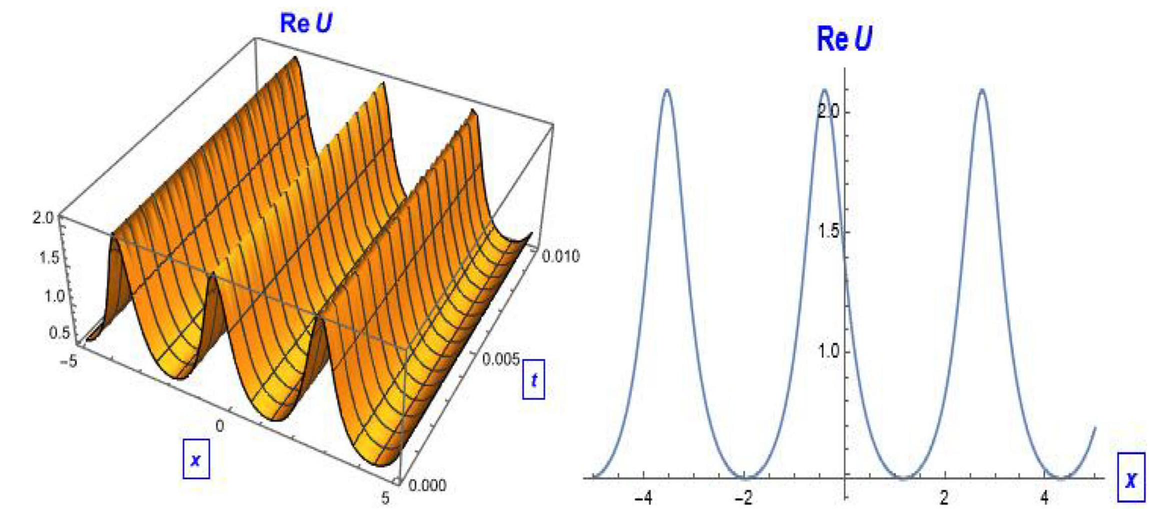

Fig. 7 The graph design of ReU Eq. (60) in 2D and 3D with values

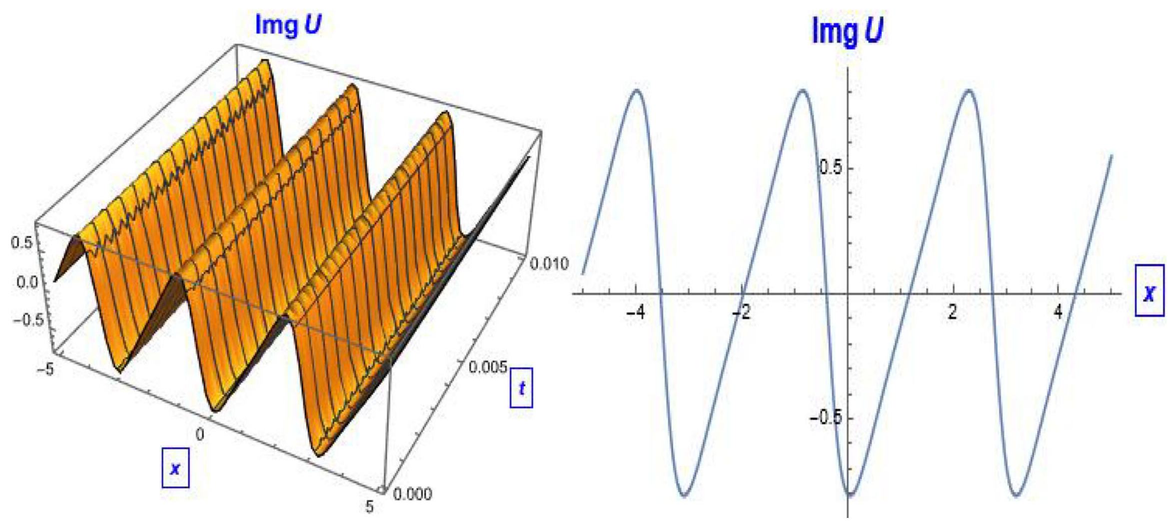

Fig. 8 The graph design of Im U Eq. (61) in 2D and 3D with values

$$A_0 = \frac{\sqrt{k-1}\sqrt{n+(k-1)w}}{\sqrt{2w\varepsilon}}, A = \frac{A_1 w\sqrt{w\varepsilon}\sqrt{n+(k-1)w}}{c(k-1)\sqrt{-w(n+(k-1)w)}}, N = \frac{-\sqrt{2}\sqrt{-w(n+(k-1)w}}{c\sqrt{k-1}} \tag{55}$$

$$A_0 = \frac{-\sqrt{k-1}\sqrt{n+(k-1)w}}{\sqrt{2w\varepsilon}}, A = \frac{A_1 w\sqrt{w\varepsilon}\sqrt{n+(k-1)w}}{c(k-1)\sqrt{-w(n+(k-1)w)}}, N = \frac{\sqrt{2}\sqrt{-w(n+(k-1)w}}{c\sqrt{k-1}} \tag{56}$$

$$A_0 = \frac{\sqrt{k-1}\sqrt{n+(k-1)w}}{\sqrt{2w\varepsilon}}, A = \frac{A_1\sqrt{w\varepsilon}\sqrt{-w(n+(k-1)w}}{c(k-1)\sqrt{n+(k-1)w}}, N = \frac{\sqrt{2}\sqrt{-w(n+(k-1)w}}{c\sqrt{k-1}} \tag{57}$$

Let us take the last result which is:

$$A_0 = \frac{\sqrt{k-1}\sqrt{n+(k-1)w}}{\sqrt{2w\varepsilon}},\ A = \frac{A_1\sqrt{w\varepsilon}\sqrt{-w(n+(k-1)w}}{c(k-1)\sqrt{n+(k-1)w}},\ N = \frac{\sqrt{2}\sqrt{-w(n+(k-1)w}}{c\sqrt{k-1}}$$

That can be simplified to be (Figs. 7, 8, 9, 10)

$$A_0 = A_1 = 1, A = i, N = 2i, \varepsilon = c = w = n = D = \chi_0 = 1 \tag{58}$$

And implement the corresponding solution to be in the form

$$U(\zeta) = 1 + \frac{e^{-2i\zeta}}{i\left(1 - \frac{e^{-2i\zeta}}{2i}\right) + 1} \tag{59}$$

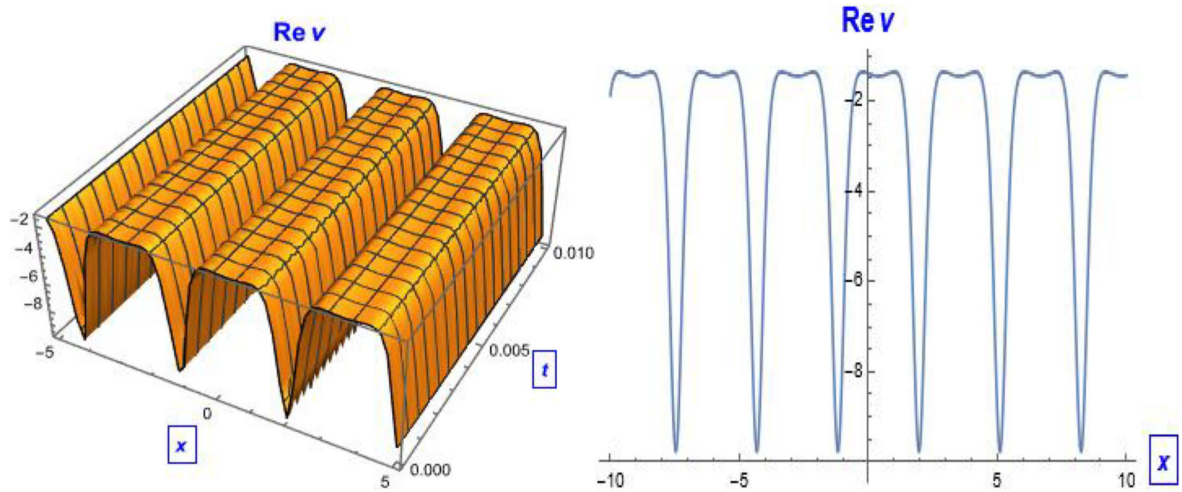

Fig. 9 The graph design of *Rev* Eq. (63) in 2D and 3D with values

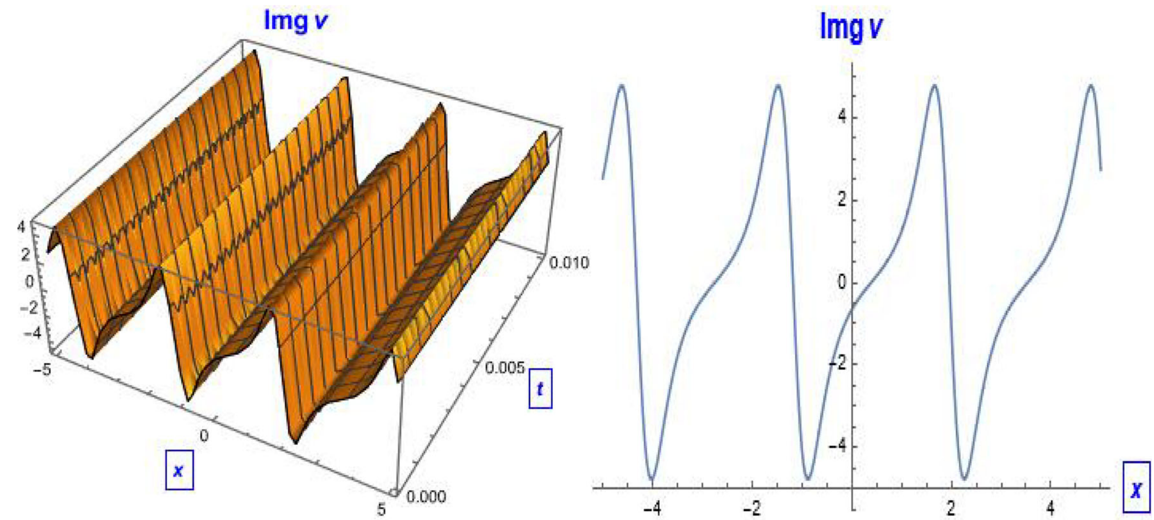

Fig. 10 The graph design of *Imv* Eq. (64) in 2D and 3D with values

$$\mathrm{Re}U(\zeta) = \frac{7}{9 + 4\cos 2\zeta + 4\sin 2\zeta} \tag{60}$$

$$\mathrm{Im}U(\zeta) = \frac{4\cos 2\zeta - 4\sin 2\zeta}{9 + 4\cos 2\zeta + 4\sin 2\zeta} \tag{61}$$

$$v(x,t) = -2\left\{\frac{7}{9 + 4\cos 2\zeta + 4\sin 2\zeta} + i\frac{4\cos 2\zeta - 4\sin 2\zeta}{9 + 4\cos 2\zeta + 4\sin 2\zeta}\right\}^2 - 1 \tag{62}$$

$$\mathrm{Re}\ v(x,t) = -2\left(\left(\frac{7}{9 + 4\cos 2\zeta + 4\sin 2\zeta}\right)^2 - \left(\frac{4\cos 2\zeta - 4\sin 2\zeta}{9 + 4\cos 2\zeta + 4\sin 2\zeta}\right)^2\right) - 1 \tag{63}$$

$$\mathrm{Im}\ v(x,t) = \frac{-28(4\cos 2\zeta - 4\sin 2\zeta)}{(9 + 4\cos 2\zeta + 4\sin 2\zeta)^2} \tag{64}$$

$A_0 = A_1 = 1,\ A = i,\ N = 2i,\ \varepsilon = c = w = n = D = \chi_0 = 1.$
$A_0 = A_1 = 1,\ A = i,\ N = 2i,\ \varepsilon = c = w = n = D = \chi_0 = 1.$
$A_0 = A_1 = 1,\ A = i,\ N = 2i,\ \varepsilon = c = w = n = D = \chi_0 = 1.$
$A_0 = A_1 = 1,\ A = i,\ N = 2i,\ \varepsilon = c = w = n = D = \chi_0 = 1.$
We can easily implement the plot of other solutions.

4 The numerical remediation using differential transform method (DTM)

In almost nonlinear problems in chemistry, physics and engineering, it is usually hard to solve analytically, and exact solutions are rather difficult to be obtained. So, in this case, the semi-analytical and numerical methods become impressive alternative approach. The differential transform method (DTM) has been proved to be efficient for handling nonlinear problems [62–65]. The idea of differential transform was first introduced by Zhou [66] who used it to solve linear and nonlinear initial value problems in electric circuit analysis. The differential transform of the kth derivative of function $y(x)$ is defined as

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right] \tag{65}$$

where $Y(k)$ is the transformed function, and $y(x)$ is the original function. Differential inverse transform of $Y(k)$ is defined as

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k \approx y_N(x) = \sum_{k=0}^{N} Y(k)x^k \tag{66}$$

By substituting Eq. (66) in (65), we get

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\frac{d^k y(x)}{dx^k}\bigg|_{x=0} \tag{67}$$

In the previous definition, we consider the case when $x = 0$, but it is true for any fixed real number $x = x_0$. The main theorems that can be derived from Eqs. (65) and (67) can be obtained from [67–69].

4.1 DTM solution corresponding to the achieved solutions by using the first family of ESEM

In this section, we introduce the numerical treatment to construct the wave solutions of Eq. (13), according to the first family of the ESEM and the values

$$k = 2,\ c = m = n = w = 1,\ \varepsilon = -1$$

Equation (13) becomes

$$U'' - 2U - 2U^3 = 0, \tag{68}$$

From Eq. (37), the exact solution of Eq. (68) and the initial conditions are:

$$U(\zeta) = -\cot(\zeta + 1), \zeta = x + t, U(0) = -\cot(1), U'(0) = \csc^2(1) \tag{69}$$

Applying the differential transform method on the system (68–69), we get on

$$(k + 1)(k + 2)Y(k + 2) - 2Y(k) - 2F(k) = 0, \tag{70}$$

where $Y(k)$ is the transformed function corresponding to the original function $U(\xi)$, , and $F(k)$ is the transformation for the nonlinear term U^3 which given from

$$F(k) = \begin{cases} (Y(0))^3; if\, k = 0 \\ \frac{1}{Y(0)} \sum_{r=1}^{k} Y(r)F(k-r); if\, k \geq 1 \end{cases}, \tag{71}$$

From Eq. (70), we get on the recurrence relation as

$$Y(k + 2) = \frac{2Y(k) + 2F(k)}{(k + 1)(k + 2)}; Y(0) = U(0) = -cot(1), Y(1) = U\prime(0) = \csc^2(1) \tag{72}$$

Putting $k = 0,$ in Eq. (72), we get on

$$Y(2) = \frac{2Y(0) + 2F(0)}{(1)(2)} = -0.906816$$

By the same way, we can get on the coefficients $Y(k)$ as

$$Y(2) = -0.906816,\ Y(3) = 1.05302,$$

Then, the DTM solution of Eq. (68) with the initial conditions (69) is

$$U(\zeta)=\sum_{k=0}^{\infty}Y(k)\zeta^k \Rightarrow U_N(\zeta)\cong\sum_{k=0}^{N}Y(k)\zeta^k=\sum_{k=0}^{10}Y(k)\zeta^k=-0.642092$$
$$+1.4122829\zeta-0.906816\zeta^2+1.0530211\zeta^3-0.9784092\zeta^4+1.0106204\zeta^5$$
$$-0.99519981\zeta^6+1.0022733\zeta^7-0.9989472\zeta^8+1.00049349\zeta^9-0.9997700\zeta^{10} \quad (73)$$

This is very identical with Maclaurin Taylor series expansion of the exact solution

$$U_{Exact}(\zeta)=-0.642092+1.4122829\zeta-0.906816\zeta^2+1.0530211\zeta^3$$
$$-0.9784092\zeta^4+1.0106204\zeta^5-0.99519981\zeta^6+1.0022733\zeta^7$$
$$-0.9989472\zeta^8+1.00049349\zeta^9-0.9997700\zeta^{10} \quad (74)$$

And the absolute error between the exact solution and numerical solution using the DTM is:

$$Error=|Exact\ solution-DTM\ solution|=\begin{bmatrix}0\\0\\5.55\times10^{-17}\\1.665\times10^{-16}\\1.665\times10^{-16}\\4.1\times10^{-17}\\2.22\times10^{-16}\\3.469\times10^{-16}\\3.053\times10^{-16}\\6.1\times10^{-16}\\1.55\times10^{-15}\end{bmatrix} \quad (75)$$

4.2 DTM solution corresponding to the achieved solutions by using the second family of ESEM

In this section, we compare between numerical treatment to the wave solutions of Eq. (13), and the exact solution of the same equation according to the second family of the ESEM.

Using the values

$$k=2,\ c=m=w=1,\ \varepsilon=-1,\ n=-3$$

Equation (13) becomes

$$U''+2U-2U^3=0, \quad (76)$$

From Eq. (48), the exact solution of Eq. (76) with $\zeta_0=0.3$ and the initial conditions are:

$$U(\xi)=\frac{1+Exp[2(\zeta+\zeta_0)]}{1-Exp[2(\zeta+\zeta_0)]}.\xi=x+t,U(0)=\frac{1+e^{0.6}}{1-e^{0.6}}=-3.43274,U\prime(0)=10.7837 \quad (77)$$

Applying the differential transform method on the system (76–77), we get on

$$(k+1)(k+2)Y(k+2)+2Y(k)-2F(k)=0, \quad (78)$$

where $Y(k)$ the transformed function corresponding to the original functions $U(\xi)$, and $F(k)$ is the transformation for the nonlinear term U^3 given in Eq. (71).

From Eq. (78), the recurrence relation is

$$Y(k+2)=\frac{-2Y(k)+2F(k)}{(k+1)(k+2)};Y(0)=U(0)=-3.43274,Y(1)=U\prime(0)=10.7837 \quad (79)$$

Putting $k=0$, in Eq. (79) we get on

$$Y(2)=\frac{-2Y(0)+2F(0)}{(1)(2)}=-37.0176$$

By the same way, we can get on the coefficients $Y(k)$ and then the DTM solution of Eq. (76) with the initial conditions (77) is:

$$U(\zeta)=\sum_{k=0}^{\infty}Y(k)\zeta^k \Rightarrow U_N(\zeta)\cong\sum_{k=0}^{N}Y(k)\zeta^k=\sum_{k=0}^{10}Y(k)\zeta^k=-3.43274$$

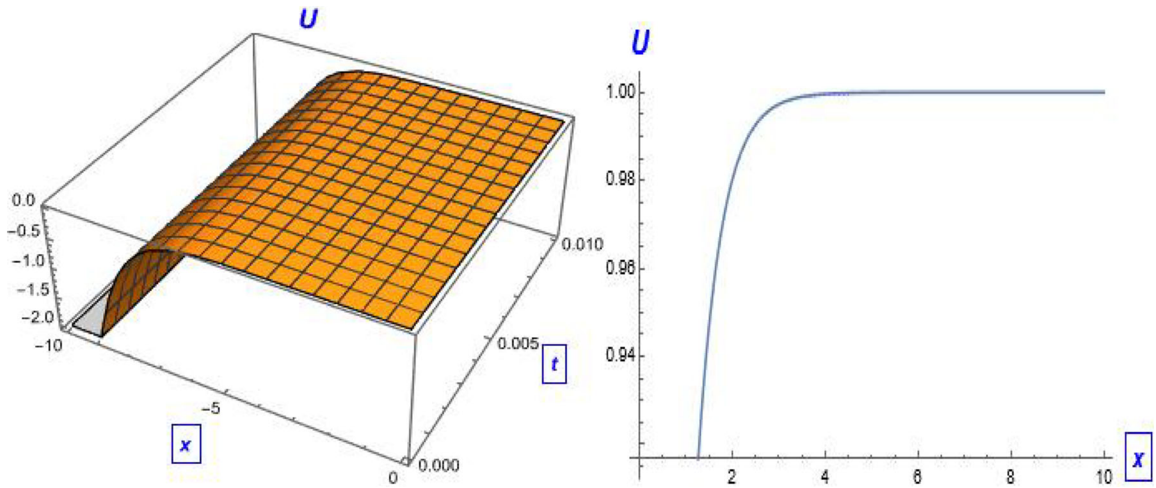

Fig. 11 The graph of DTM solution $U(x, t)$ Eq. (76) in 2D and 3D

$$+ 10.7837\zeta - 37.0176\zeta^2 + 123.477\zeta^3 - 411.526\zeta^4 + 1371.74\zeta^5 - 4572.47\zeta^6 + 15241.6\zeta^7 - 50805.3\zeta^8 + 169351\zeta^9 - 564503\zeta^{10} \tag{80}$$

This is very identical with Maclaurin Taylor series expansion of the exact solution

$$U_{Exact}(\zeta) = -3.43274 + 10.7837\zeta - 37.0176\zeta^2 + 123.477\zeta^3 - 411.526\zeta^4 + 1371.74\zeta^5 - 4572.47\zeta^6 + 15241.6\zeta^7 - 50805.3\zeta^8 + 169351\zeta^9 - 564503\zeta^{10} \tag{81}$$

And the absolute error between the exact solution and numerical solution using the DTM is (Fig. 11):

$$Error = |Exact\ solution - DTM\ solution| = \begin{bmatrix} 0 \\ 0 \\ 4.44 \times 10^{-16} \\ 1.33 \times 10^{-15} \\ 7.1 \times 10^{-15} \\ 1.13 \times 10^{-13} \\ 9.09 \times 10^{-13} \\ 5.45 \times 10^{-12} \\ 1.45 \times 10^{-11} \\ 8.73 \times 10^{-11} \\ 2.32 \times 10^{-10} \end{bmatrix} \tag{82}$$

4.3 DTM solution corresponding to the solution achieved by using the PPAM

Now, we compare between numerical results obtained to the wave solutions of Eq. (13) and the exact solution of the same equation using PPAM.

Using the values

$$k = 2, \varepsilon = c = m = n = w = 1,$$

Equation (13) becomes

$$U'' - 2U + 2U^3 = 0, \tag{83}$$

This has exact solution given in Eq. (59) and its initial conditions given as

$$U(\zeta) = 1 + \frac{e^{-2i\zeta}}{i\left(1 - \frac{e^{-2i\zeta}}{2i}\right) + 1}; \xi = x + t, U(0) = \frac{3 + 2i}{1 + 2i}; U\prime(0) = -\frac{2i}{1 + 2i} - \frac{2i(3 + 2i)}{(1 + 2i)^2} \tag{84}$$

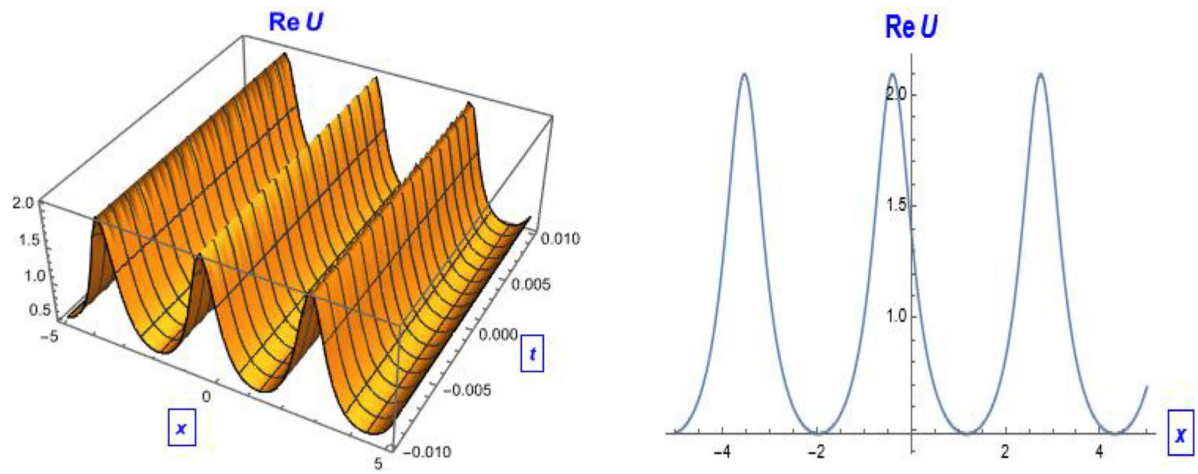

Fig. 12 The graph design of *Real*(U) using DTM Eq. (87) in 2D and 3D

Applying the differential transform method on the system (83–84), we get on

$$(k+1)(k+2)Y(k+2) - 2Y(k) + 2F(k) = 0, \tag{85}$$

where $Y(k)$ is the transformed function corresponding to the original functions $U(\xi)$, , and $F(k)$ is the transformation for the nonlinear term U^3 given in Eq. (71).

From Eq. (85), we get on the recurrence relation

$$Y(k+2) = \frac{2Y(k) - 2F(k)}{(k+1)(k+2)}; Y(0) = U(0) = \frac{3+2i}{1+2i},$$
$$Y(1) = U\prime(0) = -\frac{2i}{1+2i} - \frac{2i(3+2i)}{(1+2i)^2} \tag{86}$$

Putting $k = 0,$ in Eq. (86), we get on

$$Y(2) = \frac{2Y(0) - 2F(0)}{(1)(2)} = 1.344 + 3.392i$$

By the same way, we can get on the coefficients $Y(k)$ and then the DTM solution of Eq. (83) with the initial conditions (84) is:

$$\begin{aligned} U(\zeta) = \sum_{k=0}^{\infty} Y(k)\zeta^k \Rightarrow U_N(\zeta) \cong \sum_{k=0}^{N} Y(k)\zeta^k = \sum_{k=0}^{10} Y(k)\zeta^k = (1.4 - 0.8i) - (2.24 + 0.32i)\zeta \\ + (1.344 + 3.392i)\zeta^2 + (2.92693 - 4.70187i)\zeta^3 - (8.47616 - 0.794453i)\zeta^4 \\ + (8.81903 + 9.65666i)\zeta^5 + (3.66858 - 19.7318i)\zeta^6 - (27.5594 - 13.7631i)\zeta^7 \\ + (42.3532 + 21.0202i)\zeta^8 - (13.4391 + 71.3184i)\zeta^9 - (74.9129 - 82.44i)\zeta^{10} \end{aligned} \tag{87}$$

This is the same result of Maclaurin Taylor series expansion of the exact solution

$$\begin{aligned} U_{Exact}(\zeta) = (1.4 - 0.8i) - (2.24 + 0.32i)\zeta \\ + (1.344 + 3.392i)\zeta^2 + (2.92693 - 4.70187i)\zeta^3 - (8.47616 - 0.794453i)\zeta^4 \\ + (8.81903 + 9.65666i)\zeta^5 + (3.66858 - 19.7318i)\zeta^6 - (27.5594 - 13.7631i)\zeta^7 \\ + (42.3532 + 21.0202i)\zeta^8 - (13.4391 + 71.3184i)\zeta^9 - (74.9129 - 82.44i)\zeta^{10} \end{aligned} \tag{88}$$

And the absolute error between the exact solution and numerical solution using the DTM is given from Eq. (87 & 88) (Figs. 12, 13).

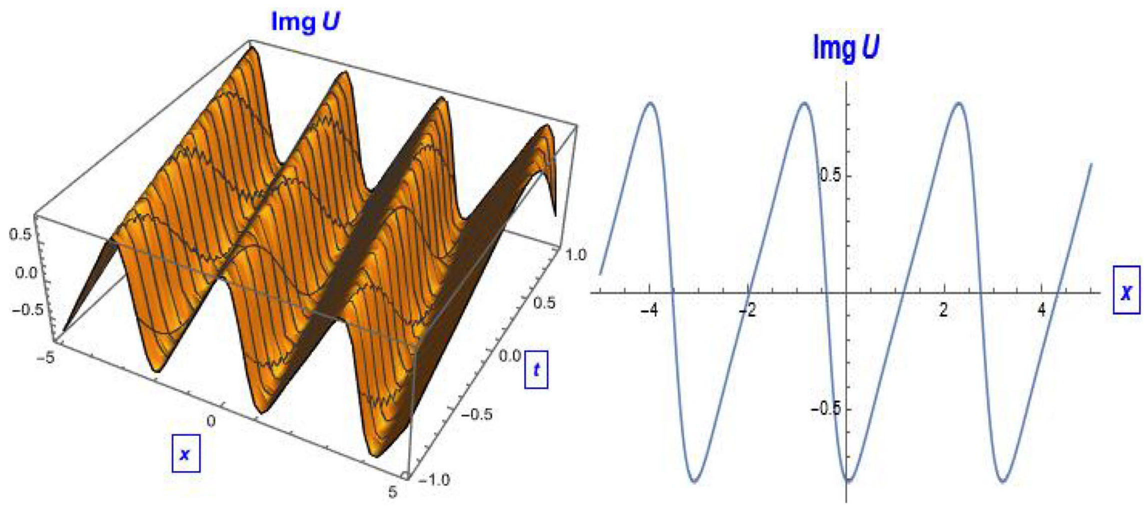

Fig. 13 The graph design of Imaginary (U) using DTM Eq. (87) in 2D and 3D

$$Error = |Exact\ solution - DTM\ solution| = \begin{bmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{bmatrix} \tag{89}$$

5 Result and discussion

Through our study, we discussed the IKE which is one of the famous and impressive integrable equations that related to the integrable space curves motion via the two suggested semi-analytical methods which are the ESEM, the PPAM and the numerical DTM. New types of the soliton solutions in forms periodic trigonometric function soliton solutions, parabolic function soliton solutions, singular soliton solutions, W-like soliton solutions, M-like soliton solutions have been documented. The suggested techniques are used for the first time for this target. The agreement between the obtained soliton solutions by the two suggested semi-analytical techniques as well as the identical numerical solutions realized by the suggested numerical method is cleared. Few forms of obtained solutions are consistent with that realized before with who studied this model by using various techniques [56–61], and the majority are new. The new visions of the soliton arising through our discussion are beneficial for all related phenomena of the suggested model.

6 Conclusion

In our current study, two effective algorithms are used to construct new diverse types of soliton solutions to the integrable Kuralay equation to investigate the integrable motion of space curves induced by these equations. It is also expected that we could have a clear picture about these new soliton solutions to this model. These new soliton solutions have been established by using the ESEM, the PPAM. Furthermore, the numerical solutions for all realized soliton solutions have been established in the framework of the differential transform method whose initial conditions are derived from the achieved solutions. Some of the achieved solutions in forms periodic trigonometric function soliton solutions, parabolic function soliton solutions, singular soliton solutions, W-like soliton solutions, M-like soliton solutions agreement with that realized before by other authors [56–61] who studied this model by using different manners and the majority are new. The agreement between the achieved soliton and numerical solutions has been shown. Moreover, the stability of our achieved solutions is clear when we compared between successive ratios of the obtained errors. The potentially, unlikely set of the achieved traveling wave solutions to the integrable Kuralay equation have been illustrated. All realized solutions are new and weren't achieved before and will introduce diverse future studies for many effective phenomena that are arising in different fields such as ferromagnetic materials, nonlinear optics and optical fibers.

Data Availability Statement The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References:

1. A. Bekir, F. Uygun, Exact travelling wave solutions of nonlinear evolution equations by using the (G′ /G)-expansion method. Arab J. Math. Sci. **18**(1), 73–80 (2012)
2. A. Bekir, M.S.M. Shehata, E.H.M. Zahran, Comparison between the new exact and numerical solutions of the Mikhailov-Novikov-Wang equation. Numer. Method Partial Differ. Equ. J. (2021). https://doi.org/10.1002/num.22775
3. E.H.M. Zahran, A. Bekir, Unexpected configurations for the optical solitons propagation in lossy fiber system with dispersion terms effect. Math. Meth. Appl. Sci. **46**, 1–15 (2022)
4. E.H.M. Zahran, A. Bekir, New unexpected behavior to the soliton arising from the geophysical Korteweg-de Vries equation. Mod. Phys. Lett. B **36**, 2150623 (2022)
5. E.H.M. Zahran, A. Bekir, New private types for the cubic-quartic optical solitons in birefringent fibers in its four forms of nonlinear refractive index. Opt. Quant. Electron. **53**, 680 (2021)
6. E.H.M. Zahran, A. Bekir, Multiple accurate-cubic optical solitons to the kerr-law and power-law nonlinear Schrödinger equation without the chromatic dispersion. Opt. Quant. Electron. **54**, 14 (2022)
7. E.H.M. Zahran, A. Bekir, M.F. Alotaibi, M. Omri, H. Ahmed, New impressive behavior of the exact solutions to the Benjamin–Bona–Mahony–Burgers equation with dual power-law nonlinearity against its numerical solution. Results Phys. **29**, 104730 (2021)
8. E.H.M.H. Zahran, H. Ahmed, T. Saeed, T. Botmart, New diverse variety for the exact solutions to Keller-Segel-Fisher system. Results Phys. **35**, 105320 (2022)
9. E. H. Zahran, A. Bekir, M. S. Shehata, Abundant analytical optical soliton solutions to the integrable (3+1)-Boussinesq equations, Accepted for publication in Ocean Engineering Journal
10. E.H.M.H. Zahran, H. Ahmed, Dark-soliton behaviors arising from a coupled nonlinear Schrödinger system. Results Phys. **36**, 105459 (2022)
11. E. H. Zahran, A. Bekir, M. S. Shehata, New impressive ideal optical soliton solutions to the space and time invariant nonlinear Schrödinger equation, Accepted for publication in Ocean Engineering Journal, (2022)
12. A. Bekir, E.H.M. Zahran, Optical soliton solutions of the thin-film ferro-electric materials equation according to the Painlevé approach. Opt. Quant. Electron. **53**, 118 (2021)
13. A. Bekir, E.H.M. Zahran, Painlev´e approach and its applications to get new exact solutions of three biological models instead of its numerical solutions. Int. J. Mod. Phys. B **34**, 2050270 (2020)
14. A. Bekir, E.H.M. Zahran, New visions of the soliton solutions to the modified nonlinear Schrodinger equation. Optik **232**, 166539 (2021)
15. E.H.M. Zahran, A. Bekir, H. Ahmed, A variety of exact solutions of the (2+1)-dimensional modified Zakharov–Kuznetsov equation. Mod. Phys. Lett. B **35**(33), 2150509 (2021)
16. E.H. Zahran, M.M. Khater, Modified extended tanh-function method and its applications to the Bogoyavlenskii equation. Appl. Math. Model. **40**, 769–775 (2016)
17. E.H.M. Zahran, A. Bekir, New unexpected explicit optical soliton solutions to the perturbed Gerdjikov–Ivanov equation. J. Opt. (2022). https://doi.org/10.1007/s12596-022-00928-7
18. A. Bekir, M.S.M. Shehata, E.H.M. Zahran, New perception of the exact solutions of the 3D-Fractional Wazwaz–Benjamin–Bona–Mahony (3D-FWBBM) equation. J. Interdiscip. Math. **24**(4), 867–880 (2021). https://doi.org/10.1080/09720502.2020.1806881
19. E.H. Zahran, S.M. Mirhosseini-Alizamini, M.S.M. Shehata, H. Rezazadeh, H. Ahmad, Study on abundant explicit wave solutions of the thin-film Ferro-electric materials equation. Optical Quantum Electron. (2021). https://doi.org/10.1007/s11082-021-03296-x
20. A. Bekir, E.H.M. Zahran, Bright and dark soliton solutions for the complex Kundu–Eckhaus equation. Optik **223**, 165233 (2020)
21. A. Bekir, E.H.M. Zahran, New vision for the soliton solutions to the complex Hirota-dynamical model. Phys. Scr. **96**, 055212 (2021)
22. M.S.M. Shehata, H. Rezazadeh, E.H.M. Zahran, E. Tala-Tebue, A. Bekir, New optical soliton solutions of the perturbed Fokas–Lenells equation. Commun. Theor. Phys. **71**(11), 1275 (2019). https://doi.org/10.1088/0253-6102/71/11/1275
23. E.H.M. Zahran, A. Bekir, New impressive vision solitary wave solutions of the Bogoyavlenskii equation in comparison with its numerical solutions. Opt. Quant. Electron. **54**, 743 (2022)
24. A. Bekir, E.H.M. Zahran, New multiple-different impressive perceptions for the solitary solution to the magneto-optic waveguides with anti-cubic nonlinearity. Optik **240**, 166939 (2021)
25. M.S. Osman, A.-M. Wazwaz, An efficient algorithm to construct multi-soliton rational solutions of the (2+1)-dimensional KdV equation with variable coefficients. Appl. Math. Comput. **321**, 282–289 (2018)
26. Xu. Gui-Qiong, A.-M. Wazwaz, Bidirectional solitons and interaction solutions for a new integrable fifth-order nonlinear equation with temporal and spatial dispersion. Nonlinear Dyn. **101**(1), 581–595 (2020). https://doi.org/10.1007/s11071-020-05740-1
27. A.-M. Wazwaz, Bright and dark optical solitons of the (2+1)-dimensional perturbed nonlinear Schrödinger equation in nonlinear optical fibers. Optik – Int. J. Light Electron Optics **251**(20), 168334 (2021)
28. M.H. Bashar, S.M.R. Islam, D. Kumar, Construction of traveling wave solutions of the (2+1)-dimensional Heisenberg ferromagnetic spin chain equation. PDE Appl. Math. **4**, 100040 (2021)
29. M.M. Latha, C. Christal Vasanthi, An integrable model of (2+1)-dimensional Heisenberg ferromagnetic spin chain and soliton excitations. Phys. Scr. **89**, 065204 (2014)
30. H. Triki, A.M. Wazwaz, New solitons and periodic wave solutions for the (2+1) dimensional Heisenberg ferromagnetic spin chain equation. J. Electromagn. Waves Appl. **30**, 788–794 (2016)

31. M. Inc, A.I. Aliyu, A. Yusuf, D. Baleanu, Optical solitons and modulation instability analysis of an integrable model of (2+1)-Dimensional Heisenberg ferromagnetic spin chain equation. Superlattice. Microst. **112**, 628–638 (2017)
32. S. Kumar, R. Ram Jiwari, R.C. Mittal, J. Awrejcewicz, Dark and bright soliton solutions and computational modeling of nonlinear regularized long wave model. Nonlinear Dyn. **104**(7), 1–22 (2021)
33. M. Younis, T.A. Sulaiman, M. Bilal, S.U. Rehman, U. Younas, Modulation instability analysis optical and other solutions to the modified nonlinear Schrödinger equation. Commun. Theor. Phys. **72**, 065001 (2020)
34. X.-Y. Gao, Y.-J. Guo, W.-R. Shan, Optical waves/modes in a multicomponent inhomogeneous optical fiber via a three-coupled variable-coefficient nonlinear Schrödinger system. Appl. Math. Lett. **120**, 107161 (2021)
35. X.-Y. Gao, Y.-J. Guo, W.-R. Shan, Symbolic computation on a (2+1)-dimensional generalized variable-coefficient Boiti–Leon–Pempinelli system for the water waves. Chaos Solitons Fract. **150**, 111066 (2021)
36. D.-Y. Yang, B. Tian, Q.-X. Qu, C.-R. Zhang, S.-S. Chen, C.-C. Wei, Lax pair, conservation laws, Darboux transformation and localized waves of a variable-coefficient coupled Hirota system in an inhomogeneous optical fiber. Chaos Solitons Fract. **150**, 110487 (2021)
37. Y. Shan, B. Tian, Bilinear auto-Bäcklund transformations and soliton solutions of a (3+1)-dimensional generalized nonlinear evolution equation for the shallow water waves. Appl. Math. Lett. **122**, 107301 (2021)
38. M. Wong, B. Tian, C.-C. Hu, S.-H. Liu, Generalized Darboux transformation, solitonic interactions and bound states for a coupled fourth-order nonlinear Schrödinger system in a birefringent optical fiber. Appl. Math. Lett. **119**, 106936 (2021)
39. X.-Y. Gao, Y.-J. Guo, W.-R. Shan, Looking at an open sea via a generalized (2+1)-dimensional dispersive long-wave system for the shallow water: scaling transformations, hetero-Bäcklund transformations, bilinear forms and N solitons. Eur. Phys. J. Plus **136**, 893 (2021)
40. X.-Y. Gao, Y.-J. Guo, W.-R. Shan, Bilinear forms through the binary Bell polynomials, N solitons and Bäcklund transformations of the Boussinesq–Burgers system for the shallow water waves in a lake or near an ocean beach. Commun. Theor. Phys. **72**, 095002 (2020)
41. X.-T. Gao, B. Tian, Y. Shan, G.-H. Feng, Comment on "Shallow water in an open sea or a wide channel: Auto- and non-auto-Bäcklund transformations with solitons for a generalized (2+1)-dimensional dispersive long-wave system." Chaos Solitons Fract. **151**, 111222 (2021)
42. E.H.M. Zahran, A. Bekir, Enormous soliton solutions to a (2+1)-dimensional Heisenberg ferromagnetic spin chain equation. Chin. J. Phys. **77**, 1236–1252 (2022)
43. M.H. Bashar, S.M.R. Islam, Exact solutions to the (2+1)-Dimensional Heisenberg ferromagnetic spin chain equation by using modified simple equation and improve F-expansion methods. Phys. Open **5**, 100027 (2020)
44. C.C. Vasanthi, M.M. Latha, Heisenberg ferromagnetic spin chain with bilinear and biquadratic interactions in (2+1) dimensions. Commun. Nonlinear Sci. Numer. Simulat. **28**, 109–122 (2015)
45. A.R. Seadawy, Stability analysis for Zakharov-–uznetsov equation of weakly nonlinear ion-acoustic waves in a plasma. Comput. Math. Appl. **67**, 172–180 (2014)
46. A.R. Seadawy, K.K. Ali, R.I. Nuruddeen, A variety of soliton solutions for the fractional Wazwaz–Benjamin–Bona–Mahony equations. Results Phys. **12**, 2234 (2019)
47. A.R. Seadawy, S.T.R. Rizvi, I. Ali, M. Younis, M.M. Kashif Ali, A.A. Makhlouf, Conservation laws, optical molecules, modulation instability and Painlevé analysis for the Chen–Lee–Liu model. Optical Quantum Electron. (2021). https://doi.org/10.1007/s11082-021-02823-0
48. A.R. Seadawy, M. Arshad, D. Lu, The weakly nonlinear wave propagation theory for the Kelvin–Helmholtz instability in magnetohydrodynamics flows. Chaos Solitons Fractals **139**, 11014 (2020)
49. A. Jhangeer, H. Rezazadeh, A.R. Seadawy, A study of travelling, periodic, quasiperiodic and chaotic structures of perturbed Fokas–Lenells model. Pramana J. **95**, 1 (2021)
50. M. Iqbal, A.R. Seadawy, D. Lu, Applications of nonlinear longitudinal wave equation in a magneto-electro-elastic circular rod and new solitary wave solutions. Mod. Phys. Lett. B **33**(18), 1950210 (2019)
51. J. Wang, K. Shehzad, A.R. Seadawy, M. Arshad, F. Asmat, Dynamic study of multi-peak solitons and other wave solutions of new coupled KdV and new coupled Zakharov–Kuznetsov systems with their stability. J. Taibah Univ. Sci. **17**(1), 2163872 (2023)
52. N. Nasreen, J. Muhammad, A. Jhangeer, U. Younas, Dynamics of fractional optical solitary waves to the cubic–quintic coupled nonlinear Helmholtz equation. Partial Differ. Equ. Appl. Math. **11**, 100812 (2024)
53. U. Younas, J. Muhammad, N. Nasreen, A. Khan, T. Abdeljawad, On the comparative analysis for the fractional solitary wave profiles to the recently developed nonlinear system. Ain Shams Eng. J. **16**, 102971 (2024)
54. J. Muhammad, U. Younas, N. Nasreen, A. Khan, T. Abdeljawad, Multicomponent nonlinear fractional Schrödinger equation: on the study of optical wave propagation in the fiber optics. Partial Differ. Equ. Appl. Math. **11**, 100805 (2024)
55. N. Nasreen, D. Lu, U. Younas, A.R. Seadawy, M. Iqbal, Dynamics of optical pulses with the effect of second-order spatiotemporal dispersion. Optical Quantum Electron. **56**, 852 (2024)
56. U. Younas, F. Yao, N. Nasreen, A. Khan, T. Abdeljawad, On the dynamics of soliton solutions for the nonlinear fractional dynamical system: application in ultrasound imaging. Results Phys. **57**, 107349 (2024)
57. Z. Sagidullayeva, K. Yesmakhanova, N. Serikbayev, G. Nugmanova, K. Yerzhanov, R. Myrzakulov, Integrable generalized Heisenberg ferromagnet equations in 1+1 dimensions: reductions and gauge equivalence, (2022) arXiv:2205.02073.
58. Z. Sagidullayeva, G. Nugmanova, R. Myrzakulov, N. Serikbayev, Integrable Kuralay equations: geometry, solutions and generalizations, (2022) arXiv:2206.05348.
59. Z. Sagidullayeva, K. Yesmakhanova, G. Nugmanova, R. Myrzakulov, Soliton solutions of the Kuralay equation via Hirota bilinear method. Proceedings of the 6th NMMP-2022, FAMU, Tallahassee, FL, USA, 17–19 (2022)
60. Z. Myrzakulova, G. Nugmanova, K. Yesmakhanova, N. Serikbayev, R. Myrzakulov, Integrable generalized Heisenberg ferromagnet equations with self-consistent potentials and related Yajima–Oikawa type equations, (2015). arXiv:1301.1649.
61. Z. Myrzakulova, G. Nugmanova, K. Yesmakhanova, R. Myrzakulov, Integrable motion of anisotropic space curves and surfaces induced by the Landau-Lifshitz equation, (2022). arXiv:2202.00748.
62. Z. Sagidullayeva, G. Nugmanova, R. Myrzakulov, N. Serikbayev, (2022) Integrable Kuralay equations: geometry, solutions and generalizations. Symmetry **14**, 1374 (2022)
63. C. Chen, S. Ho, Solving Partial Differential Equations by two-dimensional differential transform method. Appl. Math. Comput. **106**, 171–179 (1999)
64. F. Ziyaee, A. Tari, Differential transform method for solving two-dimensional Fredholm integral equation. Appl. Appl. Math. J. **10**(2), 14 (2015)
65. M. Kajani, N. Shehni, Differential transform method: an effective tool for solving nonlinear Volterra integro-differential equations. Aust. J. Basic Appl. Sci. **5**(9), 30–39 (2011)
66. F. Karakoç, H. Bereketoglu, Solutions of delay differential equations by using differential transform method. Int. J. Comput. Math. **86**(5), 914–923 (2009)
67. J. Zhou, *Differential transformation and its applications for electrical circuits* (Borneo Huazhong University Press, Wuhan, China, 2010)

68. R.A. Ibrahim, S. Mouhamed, Application of differential transform method with Adomian polynomial for solving RLC circuits problems and higher order differential equations. Eng. Res. J. **5**, 4 (2022)
69. Y.F. Patel, J.M. Dhodiya, One-Dimensional Differential Transform Method, in *Applications of Differential Transform to Real World Problems*. ed. by Y.F. Patel, J.M. Dhodiya (Chapman and Hall/CRC, Boca Raton, 2022), pp.1–45. https://doi.org/10.1201/9781003254959-1